

Radiation reaction and energy-momentum conservation

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Abstract We discuss subtle points of the momentum balance for radiating particles in flat and curved space-time. An instantaneous balance is obscured by the presence of the Schott term which is a finite part of the bound field momentum. To establish the balance one has to take into account the initial and final conditions for acceleration, or to apply averaging. In curved space-time an additional contribution arises from the tidal deformation of the bound field. This force is shown to be the finite remnant from the mass renormalization and it is different both from the radiation recoil force and the Schott force. For radiation of non-gravitational nature from point particles in curved space-time the reaction force can be computed substituting the retarded field directly to the equations of motion. Similar procedure is applicable to gravitational radiation in vacuum space-time, but fails in the non-vacuum case. The existence of the gravitational quasilocal reaction force in this general case seems implausible, though it still exists in the non-relativistic approximation. We also explain the putative antidamping effect for gravitational radiation under non-geodesic motion and derive the non-relativistic gravitational quadrupole Schott term. Radiation reaction in curved space of dimension other than four is also discussed

1 Introduction

One of the major tasks of gravitational wave astronomy is the precise theoretical prediction and observational measurement of gravitational waveforms from an inspiral fall of compact bodies into the supermassive black hole. This requires knowledge of the orbits with account for gravitational radiation reaction. Radiation gives rise to the reaction force which can be incorporated into the equations of motion. The standard strategy to get the reaction force consists in substitution of the retarded field produced by the body into its equation of motion. The resulting equation is believed

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to give a correct description of the instantaneous effect of radiation on the motion of the body. If the effect of reaction is small with respect to the main force, one can treat it adiabatically. But when the reaction force is not small, the situation is more subtle: the balance equations involve not only the kinetic energy-momentum of the body and that of radiation, but also a variable contribution of the bound field. Generically, an instantaneous loss of the energy-momentum by the body is not equal to the energy-momentum carried away by radiation.

This situation looks particularly simple in the flat space Maxwell-Lorentz electrodynamics, where due to linearity of the equations one can use the notion of the point-like particle described by the delta-function. The Lorentz-Dirac equation obtained by substituting the retarded field into the equation of motion and performing the mass renormalization is expected to describe the motion of the particle with account for radiation loss. However, it turns out that the momentum loss due to radiation gives only a part of the reaction force. Although the initial system of the charges and the Maxwell field obey the overall momentum conservation equations, the Lorentz-Dirac equation, treated as the particle equation of motion, violates the naively expected balance between the particle momentum and the momentum of radiation. The difference is given by the so-called Schott term, which is the third order total derivative of the coordinate. A careful comparison of the Dirac derivation [1, 2] and the Rohrlich analysis [3, 4, 5] has led Teitelboim [6] (see also [7]) to interpret the Schott term as the finite contribution of the bound (non-radiated) momentum of the charge remaining after the mass renormalization (an explicit proof of this claim was given in [8]). This may look strange, since we are used to think about the bound field as a stable Coulomb coat which is spherically symmetric in the instantaneous rest frame or pancake shaped for a relativistic charge. But this simple picture is valid only for constant velocity. Once acceleration is non-zero, the energy-momentum carried by the Coulomb coat becomes variable. The most surprising fact is that this momentum is simply proportional to the acceleration, and thus its derivative (the force) is the third derivative of the particle coordinate. Of course, the split of the total field into radiation and the bound field has to be done at any distance from the charge, not only in the wave zone. This was given by Rohrlich [5], for more recent discussion see the book by B. Kosyakov [9]. It is worth noting that in this case the finite part of the reaction force is entirely given by the time-antisymmetric part of the particle field (half-difference of the retarded and advanced potentials).

Therefore the energy-momentum balance of the system consisting of the accelerated charge and its Maxwell field includes three, but not just two, ingredients: the particle momentum, the momentum carried by radiation, and the bound electromagnetic momentum. The radiation momentum can be extracted both from the particle momentum and indirectly from the bound momentum. This explains the origin of radiation of the uniformly accelerated charge, in which case the total reaction force is zero and thus the kinetic particle momentum is constant. While the charge is undergoing a constant acceleration, its bound electromagnetic momentum decreases and is transferred to radiation. Physically, however, the acceleration has to start at some moment and to finish at some moment, and during the stages of acquiring and loosing the acceleration the bound momentum is exchanged with the kinetic mo-

mentum. Therefore, the total energy-momentum loss of the charge will be equal to the momentum carried away by radiation. But an instantaneous balance is obscured by the presence of the Schott term. Another simple situation is periodic motion. Since the ambiguous Schott term is total derivative, its contribution vanishes if one integrates over the period, or, equivalently, averages over the period.

For radiation of linear fields of non-gravitational nature or of the linearized gravitational field in curved space-time the situation is more complicated because there are no local conservation laws unless the Killing symmetries are present, and because the tail term can not be found in the closed form [10, 11] (for a review see [12]). Another new feature in this case is that the reaction force contains a finite time-symmetric contribution coming from the half-sum of the retarded and advanced potentials – the force found by DeWitt-DeWitt [13] in linearized gravity and later rediscovered in the full General Relativity by Smith and Will [14] (the WWSM force). This force, however, has nothing to do with radiation and so the work done by this force is not expected to contribute into the energy balance for radiation. In the stationary space-time the total energy of the particle and the field is conserved, so one can expect that the energy balance between radiation and the particle energy loss will hold integrally or in average. In the case of axial symmetry, similar considerations apply to an associated angular momentum. An explicit proof of this balance for scalar, electromagnetic and linearized gravitational radiation in the Kerr space-time was given in [15]. Apparently this work was not properly understood and was criticized in a number of papers until 2005, when analogous calculations were performed by other people (for a review and further references see [16]).

The case of linearized gravity is not entirely similar to other linear fields in the curved background, however. In fact, linearized gravity on the *non-vacuum* background is not a consistent linear theory since the full Bianchi identities do not allow for the harmonic gauge necessary to locally disentangle the linearized Einstein equations [17]. Physically this means that the proper source of the gravitational radiation from the point particle is not just its energy-momentum tensor, but it also includes contribution from the perturbed source of the background. Thus the total source is non-local, which raises doubts that there might exist a local equation describing radiation reaction (here we mean the non-locality stronger than that of the tail term, which is still localized on the world-line of the point particle).

2 Energy-momentum balance equation

Consider the interacting system of N point charges and the Maxwell field in Minkowski space-time which is described by the coupled system of equations

$$\partial_\nu F^{\mu\nu} = 4\pi \int \sum_{a=1}^N \dot{z}^\mu \delta(x - z_a(\tau)) d\tau, \quad m_a^0 \dot{z}_a^\mu = e_a F^\mu_{\nu} \dot{z}_a^\nu. \quad (1)$$

It has $3N + \infty$ degrees of freedom, where ∞ stands for the Maxwell field. The corresponding energy-momentum conservation equation is

$$\partial_\nu (\overset{m}{T}{}^{\mu\nu} + \overset{F}{T}{}^{\mu\nu}) = 0, \quad (2)$$

where

$$\overset{m}{T}{}^{\mu\nu} = \int \sum_{a=1}^N m_a^0 \dot{z}^\mu \dot{z}^\nu \delta(x - z_a(\tau)) d\tau, \quad \overset{F}{T}{}^{\mu\nu} = \frac{1}{4\pi} \left(F^{\mu\lambda} F_\lambda{}^\nu + \frac{\eta^{\mu\nu}}{4} F^{\alpha\beta} F_{\alpha\beta} \right). \quad (3)$$

We have introduced the bare masses m_a^0 in anticipation of mass-renormalization. Since the Maxwell equation is linear, one can decompose the total field in the vicinity of any given charge e_a into the sum of the field generated by the other $N - 1$ charges, $F_{\text{ext}}^{\mu\nu}$ (regular at its location) and the retarded field of e_a , $F_{a\text{ret}}^{\mu\nu}$. In spite of the fact that the total field acting on e_a , $F_a^{\mu\nu} = F_{\text{ext}}^{\mu\nu} + F_{a\text{ret}}^{\mu\nu}$ diverges at $x^\mu = z_a^\mu$, energy-momentum conservation is ensured by the equations of motion. The required mass renormalization does not change this statement.

The field $F_{a\text{ret}}^{\mu\nu}$ describes radiation and the bound field, which both contribute to the field energy-momentum. The overall conservation equation does not distinguish between these two parts, so an additional analysis is needed. The retarded potential at a given point x of space-time depends on the world-line variables taken at the moment of proper time $s_{\text{ret}}(x)$ defined as the solution to the equation

$$R^\mu R_\mu = 0, \quad R^\mu = x^\mu - z^\mu(s_{\text{ret}}), \quad (4)$$

satisfying $x^0 > z^0$. The advanced solution to the same equation with $z^0 > x^0$ refers to the advanced proper time $s_{\text{adv}}(x)$. Introducing the invariant distance

$$\rho = v_\mu(s_{\text{ret}}) R^\mu, \quad v^\mu = \frac{dz^\mu}{ds}, \quad (5)$$

which is equal to the spatial distance $|\mathbf{R}| = |\mathbf{x} - \mathbf{z}(s_{\text{ret}})|$ between the points of emission and observation in the momentarily co-moving Lorenz frame at the time moment $x^0 = z^0(s_{\text{ret}})$, one can present the retarded potential as (we omit the index a):

$$A_{\text{ret}}^\mu(x) = \frac{ev^\mu}{\rho} \Big|_{s_{\text{ret}}(x)}. \quad (6)$$

Introduce the normalized null vector $c^\mu = R^\mu/\rho$, such that $vc = 1$, and the unit space-like vector $u^\mu = c^\mu - v^\mu$, $u^2 = -1$. The following differentiation rules then hold:

$$c_\mu = \partial_\mu s_{\text{ret}}(x), \quad \partial_\mu \rho = v_\mu + \lambda c_\mu, \quad \partial_\mu c^\nu = \frac{1}{\rho} (\delta_\mu^\nu - v_\mu c^\nu - c_\mu v^\nu - \lambda c_\mu c^\nu), \quad (7)$$

where $\lambda = \dot{\rho} = \rho(ac) - 1$. The retarded field strength will read:

$$F_{\text{ret}}^{\mu\nu} = \frac{e(\rho(ac) - 1)}{\rho^2} v^{[\mu} c^{\nu]} - \frac{e}{\rho} a^{[\mu} c^{\nu]}. \quad (8)$$

The retarded potential in Minkowski space admits a natural decomposition with respect to T-parity:

$$A_{\text{ret}}^{\mu} = A_{\text{self}}^{\mu} + A_{\text{rad}}^{\mu}, \quad (9)$$

where the radiative part $A_{\text{rad}}^{\mu} = \frac{1}{2}(A_{\text{ret}}^{\mu} - A_{\text{adv}}^{\mu})$ obeys an homogeneous wave equation, while the self part $A_{\text{self}}^{\mu} = \frac{1}{2}(A_{\text{ret}}^{\mu} + A_{\text{adv}}^{\mu})$ has a source at $x = z(s)$. One could expect that only T-symmetric A_{self}^{μ} corresponds to the *bound* field, but it is not so. For an accelerated charge the situation is more subtle.

2.1 Decomposition of the stress tensor

Constructing the energy-momentum tensor $\overset{F}{T}^{\mu\nu}$ with the retarded field $F_{\text{ret}}^{\mu\nu}$, one finds that it admits a natural decomposition:

$$\overset{F}{T}^{\mu\nu} = \overset{F}{T}_{\text{emit}}^{\mu\nu} + \overset{F}{T}_{\text{bound}}^{\mu\nu}, \quad (10)$$

where the first term is selected by its dependence on ρ as ρ^{-2} :

$$\overset{F}{T}_{\text{emit}}^{\mu\nu} = -\frac{((ac)^2 + a^2)c_{\mu}c_{\nu}}{\rho^2}, \quad (11)$$

while the second contains higher powers of ρ^{-1} :

$$\overset{F}{T}_{\text{bound}}^{\mu\nu} = \frac{a^{(\mu}c^{\nu)} + 2(ac)c^{\mu}c^{\nu} - (ac)v^{(\mu}c^{\nu)}}{\rho^3} + \frac{v^{(\mu}c^{\nu)} - c^{\mu}c^{\nu} - \eta^{\mu\nu}}{\rho^4}/2,$$

where symmetrization without 1/2 is understood.

The “emit” part (11) has the following properties:

- it is the tensor product of two null vectors c^{μ} ,
- it is traceless,
- it falls down as $|\mathbf{x}|^{-2}$ when $|\mathbf{x}| \rightarrow \infty$,
- as follows from the differentiation rules (7), it is divergence-free without assuming the validity of the equations of motion:

$$\partial_{\mu} \overset{F}{T}_{\text{emit}}^{\mu\nu} = 0. \quad (12)$$

All these features indicate that $\overset{F}{T}_{\text{emit}}^{\mu\nu}$ describes the outgoing radiation.

Since the total energy-momentum tensor including the contribution of charges is (on shell) divergence free, with account for (12) we find that

$$\partial_\mu T_{\text{bound}}^{\mu\nu} + \partial_\mu T^{\mu\nu} = 0, \quad (13)$$

so the bound field momentum can be exchanged with the particle momentum. Note, that outside the world-line the bound stress tensor is divergence-free. It is also worth noting, that Eq. (12) does not mean that there is no reaction force acting on a particle which counterbalances the emitted momentum.

Consider now the total balance of forces. The conservation of the total four-momentum (2) implies that the sum of the mechanical momentum and the momentum carried by the electromagnetic field is constant (for simplicity we do not include the external field):

$$\frac{dp_{\text{mech}}^\mu}{ds} + \frac{dp_{\text{em}}^\mu}{ds} = 0. \quad (14)$$

Here the mechanical part is proportional to the bare mass of the charge

$$p_{\text{mech}}^\mu = \int T^{\mu\nu} d\Sigma_\nu = m^0 v^\mu, \quad (15)$$

while the field part is given by

$$p_{\text{em}}^\mu = \int T^{\mu\nu} d\Sigma_\nu, \quad (16)$$

where integration of the electromagnetic stress tensor is performed over a space-like hypersurface whose choice will be specified later on. It has to be emphasized that the stress tensor of the electromagnetic field is constructed in terms of the physical retarded field. According to the above splitting, we can write

$$\frac{dp_{\text{mech}}^\mu}{ds} = f_{\text{emit}}^\mu + f_{\text{bound}}^\mu, \quad (17)$$

$$f_{\text{emit}}^\mu = - \int T_{\text{emit}}^{\mu\nu} d\Sigma_\nu, \quad f_{\text{bound}}^\mu = - \int T_{\text{bound}}^{\mu\nu} d\Sigma_\nu. \quad (18)$$

On the other hand, the derivative of the bare mechanical momentum of the charge can be found substituting the retarded field in the equation of motion. In this case it is useful to decompose the retarded field according to (9), obtaining another split of the mechanical momentum:

$$\frac{dp_{\text{mech}}^\mu}{ds} = e F_{\text{ret}}^{\mu\nu} v_\nu = e (F_{\text{self}}^{\mu\nu} + F_{\text{rad}}^{\mu\nu}) v_\nu = f_{\text{self}}^\mu + f_{\text{rad}}^\mu. \quad (19)$$

Now, somewhat unexpectedly, $f_{\text{rad}}^\mu \neq f_{\text{emit}}^\mu$ and $f_{\text{self}}^\mu \neq f_{\text{bound}}^\mu$, the difference being called the Schott term [8]:

$$f_{\text{rad}}^\mu = f_{\text{emit}}^\mu + f_{\text{Schott}}^\mu, \quad f_{\text{self}}^\mu = f_{\text{bound}}^\mu - f_{\text{Schott}}^\mu. \quad (20)$$

Clearly,

$$f_{\text{self}}^\mu + f_{\text{rad}}^\mu = f_{\text{bound}}^\mu + f_{\text{emit}}^\mu, \quad (21)$$

as expected. Note that both f_{bound}^μ and f_{self}^μ contain divergences which mutually cancel in Eq. (21).

The forces f_{self}^μ and f_{rad}^μ can be found using the Green functions [2]

$$G_{\text{self}}(Z) = \delta(Z^2), \quad G_{\text{rad}}(Z) = \frac{Z^0}{|Z^0|} \delta(Z^2), \quad (22)$$

where $Z^\mu = Z^\mu(s, s') = z^\mu(s) - z^\mu(s')$. Substituting the value of the electromagnetic field generated by the charge on its world-line one obtains

$$f^\mu(s) = 2e^2 \int Z^{[\mu}(s, s') v^{\nu]}(s') v_\nu(s) \frac{d}{dZ^2} G(Z) ds', \quad (23)$$

for both f_{self}^μ and f_{rad}^μ . Due to delta-functions, only a finite number of Taylor expansion terms in $\sigma = s - s'$ contribute to the integral. In the four-dimensional case it is sufficient to retain the terms up to σ^3 :

$$2Z^{[\mu}(s, s') v^{\nu]}(s') v_\nu(s) = \dot{v}^\mu \sigma^2 - \frac{2}{3} (\dot{v}^\mu + v^\mu \dot{v}^2) \sigma^3 + O(\sigma^4). \quad (24)$$

Taking into account that $Z^2 = \sigma^2 + O(\sigma^4)$, the leading terms in the expansions of derivatives of the Green functions will be

$$\frac{d}{dZ^2} G_{\text{self}}(Z) = \frac{d}{d\sigma^2} \delta(\sigma^2), \quad \frac{d}{dZ^2} G_{\text{rad}}(Z) = \frac{d}{d\sigma^2} \left(\frac{\sigma}{|\sigma|} \delta(\sigma^2) \right). \quad (25)$$

Regularizing the delta functions of σ^2 by point-splitting

$$\delta(\sigma^2) = \lim_{\varepsilon \rightarrow +0} \delta(\sigma^2 - \varepsilon^2) = \lim_{\varepsilon \rightarrow +0} \frac{\delta(\sigma - \varepsilon) + \delta(\sigma + \varepsilon)}{2\varepsilon}, \quad (26)$$

with a prescription that the limit should be taken after evaluating the integrals, one finds

$$f_{\text{self}}^\mu = -\frac{e^2}{2\varepsilon} a^\mu, \quad f_{\text{rad}}^\mu = \frac{2e^2}{3} (v^\mu a^2 + \dot{a}^\mu). \quad (27)$$

After the mass renormalization, $m_0 + \frac{1}{2\varepsilon} = m$ we get the Lorentz-Dirac equation

$$ma^\mu = \frac{2e^2}{3} (v^\mu a^2 + \dot{a}^\mu). \quad (28)$$

The first terms at the right hand side is equal to the derivative of the momentum carried away by radiation,

$$f_{\text{emit}}^\mu = -\frac{dp_{\text{emit}}^\mu}{ds} = \frac{2e^2}{3} a^2 v^\mu. \quad (29)$$

Its independent evaluation by integration of $T_{\text{emit}}^{\mu\nu}$ can be found in [8]. The second total derivative term is the Schott term. It is worth noting, that within the local calculation, the Schott term originates from the T-odd part of the retarded field.

2.2 Bound momentum

An explicit evaluation of the bound momentum associated with a given moment of proper time s on the particle world-line,

$$p_{\text{bound}}^\mu(s) = \int_{\Sigma(s)} T_{\text{bound}}^{\mu\nu} d\Sigma_\nu \quad (30)$$

was given in [8], which we follow here. First of all one has to choose the space-like hypersurface $\Sigma(s)$ intersecting the world line at $x^\mu = z^\mu(s)$. A convenient choice will be the hyperplane orthogonal to the world-line

$$v_\mu(s)(x^\mu - z^\mu(s)) = 0. \quad (31)$$

The integral (30) is divergent on the world line. We introduce the small length parameter ε , the radius of the 2-sphere $\partial Y_\varepsilon(s)$ (Fig. 1), defined as the intersection of the hyperplane (31) with the hyperboloid $(x - z(s))^2 = -\varepsilon^2$. We also introduce the sphere $\partial Y_R(s)$ of the large radius R defined as the intersection of $\Sigma(s)$ with the hyperboloid $(x - z(s))^2 = -R^2$. The total field momentum can be obtained as the limit $\varepsilon \rightarrow 0$, $R \rightarrow \infty$ of the integral over the domain $Y(s) \subset \Sigma(s)$ between the boundaries $\partial Y_\varepsilon(s)$ and $\partial Y_R(s)$.

Let us evaluate the variation of this quantity between the moments s_1 and s_2 of the proper time on the world-line of the charge

$$\Delta p_{\text{em}}^\mu = \int_{Y(s_2)} T^{\mu\nu} d\Sigma_\nu - \int_{Y(s_1)} T^{\mu\nu} d\Sigma_\nu. \quad (32)$$

For the bound momentum it is convenient to consider the tubes S_ε and S_R formed as sequences of the spheres $\partial Y_\varepsilon(s)$ and $\partial Y_R(s)$ on the interval $s \in [s_1, s_2]$ and to transform this quantity to

$$\Delta p_{\text{bound}}^\mu = \int_{S_R} T_{\text{bound}}^{\mu\nu} dS_\nu - \int_{S_\varepsilon} T_{\text{bound}}^{\mu\nu} dS_\nu \quad (33)$$

in view of the conservation equation for $T_{\text{bound}}^{\mu\nu}$ outside the world-line (see the remark after Eq. (13)). Here the integration elements dS_ν are directed outwards to the world-line. The contribution from the distant surface S_R vanishes if one assumes that the acceleration is zero in the limit $s \rightarrow -\infty$ [6]. This is non-trivial: though the stress tensor (12) decays as R^{-3} at spatial infinity, the corresponding flux does not vanish

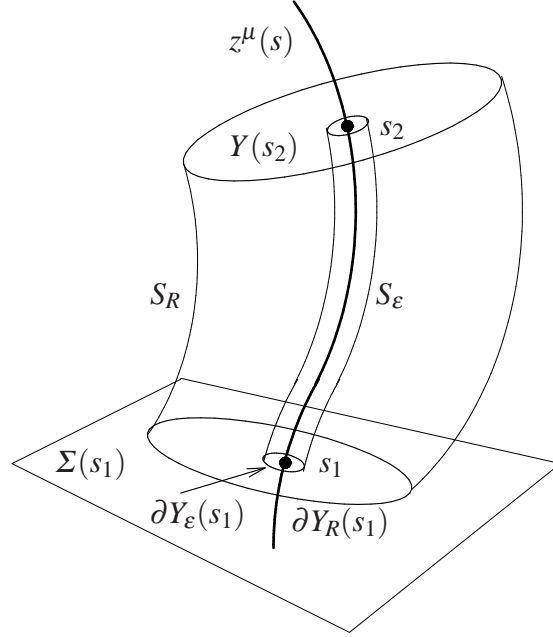


Fig. 1 Integration of the bound electromagnetic momentum. Here $\Sigma(s_1)$ is the space-like hyperplane transverse to the world-line $z^\mu(s)$ intersecting it at the proper time s_1 (similarly $\Sigma(s_2)$). The hypersurfaces S_ϵ and S_R are small and large tubes around the world-line formed by sequences of the 2-spheres $\partial Y_\epsilon(s)$ and $\partial Y_R(s)$ for $s \in [s_1, s_2]$. The domain $Y(s_2) \subset \Sigma(s_2)$ (similarly $Y(s_1)$) is the 3-annulus between $\partial Y_R(s_2)$ and $\partial Y_\epsilon(s_2)$.

a priori, because the surface element contains a term (proportional to the acceleration) which asymptotically grows as R^3 . As a consequence, the surviving term will be proportional to the acceleration taken at the moment s_{ret} of the proper time, where $s_{\text{ret}} \rightarrow -\infty$ in the limit $R \rightarrow \infty$. Finally we are left with the integral over the inner boundary S_ϵ only. To find an integration measure on S_ϵ we foliate the space-time domain shown in Fig. 1 by the hypersurfaces $\Sigma(s)$ parameterized by the spherical coordinates $r, \theta_1 = \theta, \theta_2 = \varphi$. Introducing the unit space-like vector $n^\mu(s, \theta_i)$, $n_\mu n^\mu = -1$ transverse to v^μ , we use the coordinate transformation $x^\mu = z^\mu(s) + rn^\mu(s, \theta_i)$. The induced metric on S_ϵ reads $dS_\mu = \epsilon^2[1 - \epsilon(an)]n_\mu ds d\Omega$, and hence

$$\frac{dp_{\text{bound}}^\mu}{ds} = - \int_{S_\epsilon} \epsilon^2[1 - \epsilon(an)] \tilde{T}_{\text{bound}}^{\mu\nu} n_\nu d\Omega, \quad (34)$$

where the limit $\varepsilon \rightarrow 0$ has to be taken. One has to expand $T_{\text{bound}}^{\mu\nu}$ in terms of ε . In fact, the energy-momentum tensor depends on the space-time point x^μ through the quantity ρ , depending directly on x^μ , and also through the retarded proper time s_{ret} . We have to express the resulting quantity as a function of the proper time s corresponding to the intersection of the world-line with the space-like hypersurface. We write $T_{\mu\nu}^{\text{bound}}$ in terms of the null vector $R^\mu = c^\mu \rho$:

$$\frac{4\pi}{e^2} T_{\text{bound}}^{\mu\nu} = \frac{a^{(\mu} R^{\nu)}}{\rho^4} + \frac{(2(aR) - 1) R^\mu R^\nu}{\rho^6} + \frac{(1 - (aR)) v^{(\mu} R^{\nu)}}{\rho^5} - \frac{\eta^{\mu\nu}}{2\rho^4}, \quad (35)$$

and expand R^μ as

$$R^\mu = x^\mu - z^\mu(s_{\text{ret}}) = \varepsilon n^\mu + v^\mu \sigma - \frac{1}{2} a^\mu \sigma^2 + \frac{1}{6} \dot{a}^\mu \sigma^3 + \mathcal{O}(\sigma^4), \quad (36)$$

where $\sigma = s - s_{\text{ret}} > 0$ and all the vectors are taken at s . This expansion in powers of σ has to be rewritten in terms of ε . The relation between the two can be found from the condition $R^2 = 0$:

$$\sigma = \varepsilon + \frac{an}{2} \varepsilon^2 + (9(an)^2 + a^2 - 4\dot{a}n) \frac{\varepsilon^3}{24} + \mathcal{O}(\varepsilon^4). \quad (37)$$

Substituting this into Eq. (36) and further to (35) we find:

$$\begin{aligned} \Delta p_{\text{bound}}^\mu &= \frac{e^2}{4\pi} \int_{s_1}^{s_2} ds \left\{ \frac{-n^\mu}{2\varepsilon^2} + \frac{a^\mu}{2\varepsilon} + [((an)^2 + a^2/3) v^\mu + \right. \\ &\quad \left. + ((an)^2 + a^2/2) n^\mu - 2\dot{a}^\mu/3 + 3(an)a^\mu/4 \right\} d\Omega. \end{aligned} \quad (38)$$

The leading divergent term $1/\varepsilon^2$ disappears after angular integration. Thus we obtain [8]:

$$f_{\text{bound}}^\mu = -\frac{dp_{\text{bound}}^\mu}{ds} = -\frac{e^2 a^\mu}{2\varepsilon} + \frac{2e^2}{3} \dot{a}^\mu. \quad (39)$$

Here the first divergent term has to be absorbed by the renormalization of mass, while the second is the finite Schott term. Comparing this with (27) we confirm the identity (21). Note that *a priori* the regularization parameter ε here (the radius of the small tube) is not related to the splitting parameter of the delta-function in the previous local force calculation. But actually they give the same form for divergent terms, for which reason we use the same symbol ε for both of them. With this convention, the divergent terms in the momentum conservation identity (21) mutually cancel.

The significance of the Schott term in the balance of momentum between the radiating charge and the emitted radiation is not always recognized in the text-books on classical electrodynamics. Instead, its presence is often interpreted as a drawback of classical theory, since formally it may lead to self-accelerating solutions. Mean-

while such solutions must be discriminated as unphysical since they do not satisfy proper initial/final conditions which should be imposed on the third order equation of motion [2]. From the above analysis it is clear that the Lorentz-Dirac equation, although formulated in terms of particle variables, actually describe the composite system consisting of a charge and its bound electromagnetic momentum. The redefinition of the particle momentum joining to it the bound electromagnetic momentum obscures the problem of interpretation. It is better think of the Schott term as the field degree of freedom and interpret the Lorentz-Dirac equation as the momentum balance equation for the total system including the electromagnetic field.

An instantaneous momentum balance is not just the balance between the particle and radiation, the energy-momentum can be also transferred between the particle and the field coat bound to it. Or, the radiated momentum is not always taken from the mechanical momentum of the charge, but by virtue of the Schott term, it can be extracted from the field coat too. This is what happens in the case of the uniformly accelerated charge, when the total reaction force instantaneously is zero, while radiation carries the momentum away at a constant rate. The balance is ensured by the Schott term. However, the constant acceleration can not last infinitely long. One has to consider the switching on/off processes in order to understand that finally the energy-momentum of radiation is taken from the particle. This consideration clarifies the necessity of the time averaging or integration over time needed to establish the momentum balance between radiation and the source particle. The equation of motion including the reaction force instantaneously does not imply the equality of the radiative momentum loss and the particle momentum. This feature is general enough, it is also applicable to radiation of non-gravitational nature from particles moving along the geodesics in curved space-time, as well as to the gravitational radiation.

2.3 The rest frame (non-relativistic limit)

In the rest frame of a charge the recoil force has no spatial component. This is due to the fact that radiation in two opposite directions is the same so that the spatial momentum is not lost by radiation, though the energy is lost. Hence the total spatial component of the reaction force is presented by the Schott term, namely

$$\mathbf{f}_{\text{Schott}} = \frac{2}{3}e^2\dot{\mathbf{a}}. \quad (40)$$

The work done by this force

$$\int \mathbf{f}_{\text{Schott}} \cdot \mathbf{v} dt = \int \frac{2}{3}e^2\dot{\mathbf{a}} \cdot \mathbf{v} dt = - \int \frac{2}{3}e^2\mathbf{a}^2 dt + \text{boundary terms} \quad (41)$$

correctly reproduces the radiative loss in the rest frame. (Boundary terms should vanish by appropriate asymptotic switching on/off or periodicity conditions).

3 Flat dimensions other than four

Recent interest to models with large extra dimensions motivates the study of radiation and radiation reaction in dimensions other than four. It turns out that the radiation picture is substantially different in even and odd dimensions because of the different structure of the retarded Green's functions for massless fields in the coordinate representation [18] (they still look similarly in all dimensions in the momentum representation). In even dimensions the retarded potential is localized on the past light cone (Huygens principle) so the situation is qualitatively similar to that in the 4D case. In odd dimensions it is non-zero also inside the past null cone though radiation still propagates along the null rays. In 3D, for instance, the scalar Green's function reads

$$G_{\text{ret}}^{3D}(X) = \vartheta(X^0) \vartheta(X^2) (X^2)^{(-1/2)}, \quad X^\mu = x^\mu - x'^\mu. \quad (42)$$

It does not contain the “direct” part singular on the light cone. Green's functions in higher odd dimensions $D = 2n + 1$ can be obtained by the recurrent relation [18, 19]

$$G_{\text{ret}}^{2n+1}(X) \sim \frac{dG_{\text{ret}}^{2n-1}}{dX^2} \quad (43)$$

In particular, in 5D

$$G_{\text{ret}}^{5D}(X) \sim \vartheta(X^0) \left(\frac{\delta(X^2)}{(X^2)^{1/2}} - \frac{1}{2} \frac{\vartheta(X^2)}{(X^2)^{3/2}} \right) \quad (44)$$

both the direct and the tail parts are present. It turns out that the direct part regularizes the tail contribution to the field stress proportional to the derivative of G which otherwise would be singular outside the world line.

In even dimensional space-times the split of the retarded potentials into the time symmetric and the radiative parts leads to purely divergent self-force and a finite radiative part:

$$f_{\text{self}}^\mu = f_{\text{div}}^\mu, \quad f_{\text{rad}}^\mu = \text{finite}. \quad (45)$$

Divergent terms are Lagrangian type and can be absorbed by introducing suitable counterterms. Since the Coulomb dependence is more singular at the location of the source in higher dimensions, the self-action gives rise to larger number of divergent terms $1/\varepsilon^n$, where n is changing from unity to the integer part of $D/2 - 1$. The highest divergence is absorbed by the renormalization of mass, while to absorb other divergencies the counterterms are needed depending on higher derivative of the velocity. These are not present in the initial action, so higher-dimensional classical theories are not renormalizable. In 6D one has two divergent terms (which in terms of the field split correspond to f_{self} [20]):

$$f_{\text{div}}^\mu = -\frac{1}{6\varepsilon^3} a_\mu + \frac{1}{2\varepsilon} \left(\frac{3}{4} v_\mu (\dot{a}a) + \frac{3}{8} a^2 a_\mu + \frac{1}{4} \ddot{a}_\mu \right), \quad (46)$$

the leading being eliminated by the mass renormalization and the subleading requiring the counterterm [20]

$$S_1 = -\kappa_0^{(1)} \int (\ddot{z})^2 ds, \quad (47)$$

which leads to the Frenet-Serret dynamics [21] unless the renormalized value $\kappa^{(1)} = 0$. For each two space-time dimensions one new higher-derivative counterterm is needed to absorb divergencies.

The split of the field stress-tensor built with the retarded field into the sum of the emitted and bound terms is also possible in all even dimensions, and one always has the relation (21). In 6D, e.g., the radiation recoil force in 6D is

$$f_{\text{emit}}^\mu = \frac{4}{45} e^2 \left(\dot{a}^2 v^\mu + \frac{2}{21} (a\dot{a}) a^\mu - \frac{2}{9} a^4 v^\mu - \frac{2}{105} a^2 \dot{a}^\mu \right), \quad (48)$$

and the Schott terms is

$$f_{\text{Schott}}^\mu = -\frac{4e^2}{45} \left(\ddot{a}^\mu + \frac{16}{7} a^2 \dot{a}^\mu + \frac{60}{7} (a\dot{a}) a_\mu + 4\dot{a}^2 v^\mu + 4(a\ddot{a}) v^\mu \right), \quad (49)$$

the sum of two being orthogonal to the 6-velocity. The Schott terms is again given by the finite part of the integrated bound momentum.

In odd dimensions one always have tail terms. Split of the retarded field into the self- and the radiative parts is always possible, and the substitution into the equations of motion leads to divergent and finite terms. Divergent ones can again be absorbed introducing the counterterms. The split of the stress-tensor into the emitted and the bound parts is more subtle. The situation is obscured by the fact that though the free field is massless and thus propagates along the null cone, the full retarded potential fills the interior of the past light cone. Still one is able to obtain the general formula for radiation momentum which is no more associated with the retarded proper time on the world-line [22].

4 Local method for curved space-time

An approach initiated by DeWitt and Brehme and further applied to linearized gravity in [23] appeals to computation of the integral of the stress-tensor in the world-tube surrounding the world-line. This is similar to our calculation of the bound momentum. However, in curved space-time the split of the stress tensor into the emitted and bound parts becomes problematic, so the complete analysis of the balance between radiation, the kinetic momentum and the bound momentum is not available. Meanwhile, to compute the total reaction force one can use much a simpler calculation substituting the retarded field directly into the equations of motion [17]. This approach was also formulated in higher even-dimensional space-time in the paper [19] which we follow here.

4.1 Hadamard expansion in any dimensions

As in $D = 4$ [10], the curved space Green's functions for massless fields in other dimensions can be constructed starting with the Hadamard solution. For simplicity we consider here the scalar case. The scalar Hadamard Green's function $G_H(x, x')$ is a solution of the homogeneous wave equation $\square_x G_H(x, x') = 0$, where $\square = g_{\mu\nu} \nabla^\mu \nabla^\nu$. The procedure consists in expanding $G_H(x, x')$ in terms of the Synge world function $\sigma(x, x')$. For $D = 4$ the Hadamard expansion contains two terms singular in σ , namely, σ^{-1} and $\ln \sigma$. In higher dimensions one has to add other singular terms, and by dimensionality it is easy to guess that each dimension introduces an additional factor $\sigma^{-1/2}$. Thus, the Hadamard expansion in $D = 2d$ dimensions ($d \geq 3/2$ is integer or half-integer) generically must read

$$G_H(x, x') = \frac{1}{(2\pi)^d} \left[\sum_{n=1}^D g_n \sigma^{1-n/2} + v \ln \sigma \right], \quad (50)$$

where $g_n = g_n(x, x')$, $v = v(x, x')$ are two-point functions. It can be shown, that in odd dimensions we actually have only odd powers of $\sigma^{-1/2}$, and in even dimensions — only even powers, that is, an expansion in terms of inverse integer powers of σ . The logarithmic term is present only in even dimensions.

Substituting (50) into the wave equation, in the leading singular order we will have: $g_D = \Delta^{1/2}$. In the next to leading order we obtain the equation:

$$2\partial_\mu g_{D-1} \sigma^\mu + g_{D-1} \square \sigma - (D-1)g_{D-1} = 0. \quad (51)$$

which does not have analytic solutions, so $g_{D-1} = 0$. For $D = 3$ this means the absence of the logarithmic term. Similarly, considering the equation for $g_{(D-1-2k)}$, $k \in \mathbb{N}$ we find $g_{D-1-2k} = 0$. This means that for an even-dimensional space-time the Hadamard Green's function contains only integer negative powers of σ plus logarithm and a regular part, while in the odd-dimensional case — only half-integer powers of σ plus a regular part.

For the sequence of Green's functions in the flat space-time, the one in $D + 2$ dimension is proportional to the derivative of the Green's function in the twice preceding dimension D . In fact, in even dimensions the symmetric Green's function is the derivative of the order $d - 2$ of the delta-function: $G^D \sim \delta^{d-2}(\sigma)$, $\sigma = (x - x')^2/2$ and thus, $G^{D+2} \sim dG^D/d\sigma$. Applying regularization, $\delta((x - x')^2) = \lim_{\varepsilon \rightarrow +0} \delta(|(x - x')^2| - \varepsilon^2)$, we obtain

$$G^{D+2} \sim dG^D/d\varepsilon^2. \quad (52)$$

This relation has a consequence that the Laurent expansion of the Lorentz-Dirac force in terms of ε in the even-dimensional Minkowski space has only odd negative powers, and no even terms. So the number of divergent terms in the self-action increases by one for each next even dimension. In curved space passing to the standard notation notation for g_n we have

$$G_H = \frac{1}{(2\pi)^d} \left(\sum_{k=0}^{d-2} \frac{u_k}{\sigma^{d-1-k}} + v \ln \sigma + w \right), \quad (53)$$

where $u_0 = \Delta^{1/2}$ and we denoted $v = u_{d-1}$, $w = u_d$. Applying \square with respect to x^μ , we obtain the system of recurrent differential equations for $u_i(x, x')$. Integrating them along the geodesic connecting the points x, x' , one can uniquely express $u_1(x, x')$ through $u_0(x, x')$. Furthermore, u_2 is expressed through u_1 , etc.

To relate the coefficient functions $u_i^D(x, x')$ in different dimensions we first observe that $u_0 = \Delta^{1/2}$ for any D . It is worth noting, however, that in the expansion in terms of σ

$$u_0^D = \Delta^{1/2} = 1 + 1/12 R_{\alpha\beta} \sigma^\alpha \sigma^\beta + \dots \quad (54)$$

where the tensor indices α, β run the values corresponding to D . With this in mind, we can write $u_0^{D'} = u_0^D$ for any D, D' . The next equation in the recurrence gives for $D, D' \geq 5$

$$u_1^{D'} = u_1^D \frac{d-2}{d'-2}. \quad (55)$$

Similarly for $D, \geq 7$, $D' = D+2$ one obtains:

$$u_2^{D+2} = u_2^D \frac{d-3}{d-1}. \quad (56)$$

Continuing this process further one finds

$$-\frac{1}{d-1} \frac{\partial G_H^D}{\partial \sigma} = (2\pi) G_H^{\text{dir} D+2}, \quad (57)$$

where the “direct” part of the Hadamard function is

$$G_{H \text{ dir}} = \frac{1}{(2\pi)^d} \sum_{k=0}^{d-2} \frac{u_k}{\sigma^{d-1-k}}. \quad (58)$$

4.2 Divergencies

For the retarded Green’s function one finds:

$$G_{\text{ret}} = \frac{1}{2(2\pi)^{d-1}} \Theta(x', \Sigma(x)) \left(\sum_{m=0}^{d-2} \frac{(-1)^m u_{d-2-m} \delta^{(m)}(\sigma)}{m!} - v \theta(-\sigma) \right). \quad (59)$$

The first term constitutes the direct part of the retarded function with the support on the light cone:

$$G_{\text{dir}} = \frac{1}{2(2\pi)^{d-1}} \Theta[\Sigma] \sum_{m=0}^{d-2} \frac{(-1)^m u_{d-2-m} \delta^{(m)}(\sigma)}{m!}. \quad (60)$$

Similarly to the recurrent relations of the previous section, we obtain for the retarded Green's functions:

$$\frac{\partial G_{\text{ret}}^D}{\partial \sigma} = -2\pi(d-1)G_{\text{dir}}^{D+2}. \quad (61)$$

Using the regularization $\delta(\sigma) = \lim_{\varepsilon \rightarrow +0} \delta(\sigma - \varepsilon)$, where $\varepsilon = \varepsilon^2/2$, we obtain for the direct part of the reaction force

$$f_{D+2}^{\mu \text{ dir}} = -\frac{1}{D-1} \frac{\partial f_D^{\mu \text{ dir}}(s, \varepsilon)}{\partial \varepsilon}. \quad (62)$$

The limit $\varepsilon \rightarrow +0$ has to be taken after the differentiation. The direct force is due to the light cone part of the retarded Green's function. This is not the full local contribution to the Lorentz-Dirac force. An additional contribution comes from the differentiation of the theta function in the tail term $v\theta(\sigma)$. In the scalar case this contribution vanishes, but in the electromagnetic case an extra local term arises:

$$f_{\text{loc}}^{\mu} = e^2 ([v_{\mu\alpha}] \dot{z}_\nu - [v_{\nu\alpha}] \dot{z}_\mu) \dot{z}^\nu \dot{z}^\alpha, \quad (63)$$

where the coincidence limit $[v_{\nu\alpha}]$ depends on the dimension. The remaining contribution from the tail term will have the form of an integral along the past half of the particle world line.

The direct part of the Lorentz-Dirac force contains divergences. To separate the divergent terms one can use the decomposition of the retarded potential suggested in the case of four dimensions by Detweiler and Whiting [24, 25]. In higher even dimensions we can follow essentially the same procedure. We define the ‘‘singular’’ part G_S of the retarded Green's function as the sum of the symmetric part (self) and the tail function v as follows

$$G_S(x, x') = G_{\text{self}}(x, x') + \frac{v(x, x')}{4(2\pi)^{d-1}} = G_{\text{self dir}}(x, x') + \frac{v(x, x') \theta(-\sigma)}{4(2\pi)^{d-1}}. \quad (64)$$

Here the direct part of the self function means its part without the tail v -term. The remaining part of the Green's function $G_R(x, x') = G_{\text{ret}}(x, x') - G_S(x, x')$ satisfies a free wave equation and is regular. Taking into account $\square v = 0$, it is clear that G_S satisfies the same inhomogeneous equation as G_{self} . The v -term in the second line of Eq. (64) is localized outside the light cone. Therefore the corresponding field (for instance, scalar), at an arbitrary point x will be given by

$$\phi_S(x) = \phi_{\text{self dir}} + \frac{m_0 q}{4(2\pi)^{d-1}} \int_{\tau_{\text{ret}}}^{\tau_{\text{adv}}} v(x, z(\tau)) d\tau, \quad (65)$$

where the retarded and advanced proper time values $\tau_{\text{ret}}(x)$, $\tau_{\text{adv}}(x)$ are the intersection points of the past and future light cones centered at x with the world line.

Inserting (65) into the Lorentz-Dirac force defined on the world-line $x = z(\tau)$, we observe that the integral contribution from the tail term vanishes and so the divergent part is entirely given by $\phi_{\text{self dir}}$. Using for delta-functions the point-splitting regularization we find that all divergent terms arise as negative powers of ε . To prove the existence of the counterterms we consider the interaction term in the action substituting the field as the S-part of the retarded solution to the wave equation. In the scalar case we will have:

$$S_S = \frac{1}{2} \int G_S(x, x') \rho(x) \rho(x') \sqrt{g(z)g(z')} dx dx', \quad (66)$$

where a factor one half is introduced to avoid double counting when self-interaction is considered. Substituting the currents we get the integral over the world-line. Since the Green's function is localized on the light cone we expand the integrand in terms of the difference $t = \tau - \tau'$ around the point $z(\tau)$:

$$S_S \sim \int d\tau \int \sum_{k,l} B_{kl}(\tau) \delta^{(k)}(t^2 - \varepsilon^2) t^l dt. \quad (67)$$

Here the coefficients $B_{kl}(z)$ depend on the curvature, while the delta-functions are flat: $\delta^{(k)}(t^2 - \varepsilon^2)$. By virtue of parity, the integrals with odd l vanish, so only the odd inverse powers of ε will be present in the expansion. Moreover, once we know the divergent terms in some dimension D , we can obtain by differentiation all divergent terms in $D + 2$, except for $1/\varepsilon$ term. The linearly divergent term corresponds to $l = 2k$. The integral is equal to

$$\int_0^\infty \delta^{(k)}(t^2 - \varepsilon^2) t^{2k} dt = \frac{(-1)^k (2k-1)!!}{2^{k+1}} \frac{1}{\varepsilon}.$$

In four dimensions this term is unique. Applying our recurrence chain we obtain the inverse cubic divergence in six dimensions and calculate again the linearly divergent term. Thus in $D = 2d$ dimensions we will get $d - 1$ divergent terms from which $d - 2$ can be obtained by the differentiation of the previous-dimensional divergence, and the linearly divergent will be new. This linearly divergent self-action term in the action will have generically the form

$$S_S^{(-1)} \sim \frac{1}{\varepsilon} \int \sum_{k=0}^{k=?} \frac{(-1)^k (2k-1)!!}{2^{k+1}} B_{k,2k}(\tau) d\tau.$$

Here the coefficient functions are obtained taking the coincidence limits of the two-point tensors involved in the expansion of the Hadamard solution. They actually depend on the derivatives of the world line embedding function $z(\tau)$ as well as the curvature terms taken on the world line:

$$B_{k,2k}(\tau) = B_{k,2k}(\dot{z}, \ddot{z}, \dots, R(z(\tau)), R_{\mu\nu}(z(\tau)), \dots).$$

The vector case is technically the same, now one has to expand in powers t in the integrand of

$$S^S = \frac{1}{2} \int G_{\mu\alpha}^S(x, x') j^\mu(x) j^\alpha(x') \sqrt{g(z)g(z')} dx dx'.$$

From this analysis it follows that in any dimension the highest divergent term can be absorbed by the renormalization of the mass as in the generating four dimensional case. To absorb the remaining $d - 2$ divergences one has to add to the initial action the sum of $d - 2$ counterterms depending on higher derivatives of the particle velocity. Typically the counterterms depend on the Riemann tensor of the background.

4.3 Four dimensions

In four dimensions the scalar retarded Green's function contains a single direct term localized on the light cone and a tail term:

$$G_{\text{ret}} = \frac{1}{4\pi} \theta[\Sigma(x), x'] \left[\Delta^{1/2} \delta(\sigma) - v \theta(\sigma) \right]. \quad (68)$$

The retarded solution for the scalar field reads

$$\phi_{\text{ret}}(x) = m_0 q \int_{-\infty}^{\tau_{\text{ret}}(x)} \left[-\Delta^{1/2} \delta(\sigma) + v \theta(\sigma) \right] d\tau'. \quad (69)$$

Differentiating this expression we obtain $\phi_v = \partial_v \phi$ on the world line:

$$\phi_v(z(\tau)) = m_0 q \int_{-\infty}^{\tau} [-\Delta^{1/2} \delta'(\sigma) \sigma_v - \Delta_{;v}^{1/2} \delta(\sigma) - v \delta(\sigma) \sigma_v + v_v] d\tau', \quad (70)$$

where integration is performed along the past history of the particle. All the two-point functions are taken on the world-line at the points $x = z(\tau)$ (observation point) and $z' = z(\tau')$ (emission point).

To compute local contributions from the terms proportional to delta-functions and its derivative, it is enough to expand the integrand in terms of the difference $s = \Delta\tau = \tau - \tau'$ around the point $z(\tau)$. The Taylor (covariant) expansion of the fundamental biscalar $\sigma(z(\tau), z(\tau'))$ is given by [26]:

$$\sigma(z(\tau), z(\tau')) = \sum_{k=0}^{\infty} \frac{1}{k!} D^k \sigma(\tau, \tau) (\tau - \tau')^k, \quad (71)$$

where we denote as $Q(\tau, \tau')$ the quantity $Q(z(\tau), z(\tau'))$ for any Q taken on the world-line, and D is a covariant derivative along the world-line (also denoted by dot):

$$D\sigma \equiv \dot{\sigma} = \sigma_\alpha \dot{z}^\alpha, \quad D^2\sigma \equiv \ddot{\sigma} = \sigma_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta + \sigma_\alpha \ddot{z}^\alpha,$$

etc. Such an expansion exists since the difference $s = \tau - \tau'$ is a two-point scalar itself: this is the integral from the scalar function $\int (-\dot{z}^2)^{1/2} ds$, along the world-line from $z(\tau)$ to $z(\tau')$. Taking the limits and using $\dot{z}^2(\tau) = -1$, we find:

$$\sigma(s) = -\frac{s^2}{2} - \ddot{z}^2(\tau) \frac{s^4}{24} + \mathcal{O}(s^5). \quad (72)$$

To obtain an expansion of the derivative of σ over $z^\mu(\tau)$ one can expand $\sigma^\mu(\tau, \tau - s)$ in powers of s . This quantity transforms as a vector at $z(\tau)$ and a scalar at $z(\tau')$, this

$$\sigma^\mu(s) = s \left(\dot{z}^\mu - \ddot{z}^\mu \frac{s}{2} + \ddot{z}^\mu \frac{s^2}{6} \right) + \mathcal{O}(s^4), \quad (73)$$

where the index μ corresponds to the point $z(\tau)$: $\sigma^\mu = \partial\sigma(z, z')/\partial z_\mu$. Recall that the initial Greek indices correspond to $z(\tau')$. The expansion of $\delta(-\sigma)$ will read:

$$\delta(-\sigma) = \delta(s^2/2) + s^4 \frac{\ddot{z}^2(\tau)}{24} \delta'(s^2/2) + \dots \quad (74)$$

where the derivative of the delta-function is taken with respect to the full argument. Since the most singular term is $\Delta^{1/2} \delta'(\sigma) \sigma_\nu$, the maximal order giving the non-zero result after the integration is s^3 . (Note, that in order to use our dimensional recurrent relations to obtain a reaction force in higher dimensions we should perform an expansion up to higher orders in s .) Thus, with the required accuracy, $\delta(-\sigma) = 2\delta(s^2)$, and all the integrals for the delta-derivatives are the same as in the flat space-time. This allows us to use the same regularization of the delta-functions with double roots $\delta(s^2)$ by the point-splitting. Expanding the biscalar $\Delta^{1/2}$ and its gradient at z we have:

$$\Delta^{1/2} = 1 + \frac{s^2}{12} R_{\sigma\tau} \dot{z}^\tau \dot{z}^\sigma, \quad \partial_\nu \Delta^{1/2} = \frac{s}{6} R_{\nu\tau} \dot{z}^\tau. \quad (75)$$

Combining all the contributions we obtain finally for the field strength on the world-line:

$$\phi_\nu \Big|_{z(\tau)} = m_0 q \left(\frac{1}{2\epsilon} \ddot{z}_\nu - \frac{1}{3} \ddot{z}_\nu - \frac{1}{6} R_{\nu\tau} \dot{z}^\tau - \frac{1}{6} R_{\gamma\delta} \dot{z}^\gamma \dot{z}^\delta \dot{z}_\nu + \frac{1}{12} R \dot{z}_\nu + \int_{-\infty}^{\tau} \nu_\nu d\tau' \right). \quad (76)$$

After the renormalization of mass one recovers the familiar equation.

Similarly, in the electromagnetic case one finds the retarded Maxwell tensor on the world line $x = z(\tau)$:

$$F_{\mu\nu}^{\text{ret}} \Big|_{z(\tau)} = e \int_{-\infty}^{\tau} [u_{\mu\alpha;\nu} \delta(\sigma) + u_{\mu\alpha} \sigma_{\nu} \delta'(\sigma) + v_{\nu\alpha;\mu} + v_{\mu\alpha} \sigma_{\nu} \delta(\sigma) - \{\mu \leftrightarrow \nu\}] \dot{z}^{\alpha} d\tau'. \quad (77)$$

Performing expansions on the world-line, one easily recover the DeWitt-Brehme-Hobbs result

$$f_{\text{em}}^{\mu} = e^2 \left[-\frac{\ddot{z}^{\mu}}{2\varepsilon} + \Pi^{\mu\nu} \left(\frac{2}{3} \ddot{z}_{\nu} + \frac{1}{3} R_{\nu\alpha} \dot{z}^{\alpha} \right) + \dot{z}^{\nu}(\tau) \int_{-\infty}^{\tau} (v_{\alpha;\nu}^{\mu} - v_{\nu\alpha}^{\mu}) \dot{z}^{\alpha}(\tau') d\tau' \right], \quad (78)$$

where $\Pi^{\mu\nu} = g^{\mu\nu} - \dot{z}^{\mu} \dot{z}^{\nu}$. For geodesic motion the entire reaction force is given by the tail term.

4.4 Self and rad forces in curved space-time

Apart from the tail term, another new feature of the instantaneous momentum balance between the radiating charge and radiation is the presence of the finite contribution in the self part of the reaction force originating from the half-sum of the retarded and advanced potentials. As we have seen in the previous section, in the case of Minkowski space the self part in four and other even dimensions is a pure divergence which has to be absorbed by renormalization of the mass and (in higher dimensions) the bare coupling constants in the higher derivative counterterms. In curved space, as was first shown in [10], the tail term in the equation for radiating charge moving along the geodesic with non-relativistic velocity (in weak gravitational field) contains apart from the dissipative term also the conservative force. This conservative force was later found for a static charge in the Schwarzschild metric [14]. Here we would like to explore the significance of this result in view of the above analysis of self/rad decomposition. In the paper [13] the splitting of the retarded field into the self and rad parts was not used. Considering the geodesic motion we have the equation:

$$m \ddot{z}^{\alpha} = e^2 \dot{z}^{\beta} \int_{-\infty}^{\tau} v_{\text{ret}\beta\gamma}^{\alpha} \dot{z}^{\gamma}(s') ds'. \quad (79)$$

Splitting the tail function with respect to T-parity

$$v_{\text{ret}\beta\gamma}^{\alpha} \dot{z}^{\gamma} = v_{\text{self}\beta\gamma}^{\alpha} \dot{z}^{\gamma} + v_{\text{rad}\beta\gamma}^{\alpha} \dot{z}^{\gamma}, \quad (80)$$

and repeating the calculations along the lines of [13] one finds in the weak-field non-relativistic case the corresponding (spatial) parts of the reaction force in terms of the flat space theory:

$$\mathbf{f}_{\text{self}} = \mathbf{f}_{\text{div}} + \mathbf{f}_{\text{WWSM}}, \quad \mathbf{f}_{\text{rad}} = \mathbf{f}_{\text{Schott}}, \quad (81)$$

where the divergent term is the same as in the flat space. The two finite terms

$$\mathbf{f}_{\text{WWSM}} = \frac{GMe^2}{r^4} \mathbf{r}, \quad \mathbf{f}_{\text{Schott}} = \frac{2}{3} e^2 \dot{\mathbf{a}} \quad (82)$$

are the WWSW force and the Schott force. Therefore, the WWSW force is a finite part of the T-even (self) contribution to the tail, while in the T-odd (rad) part reproduces the Schott term which is precisely the same as in the flat space. Similar result holds for the scalar radiation.

5 Gravitational radiation

Gravitational radiation in the framework of linearized gravity on the curved background may seem similar to scalar or electromagnetic radiation, but the similarity is incomplete. The difference is that radiation of non-gravitational nature emitted by bodies moving along geodesics in the fixed background is not influenced by the non-linear nature of the Einstein equations. For gravitational radiation this is not so. The scalar or vector linear field equations in curved space imply vanishing of the covariant divergence of the field stress tensor of matter field with respect to the background metric. In the gravitational case one has the Bianchi identity which generally does not imply the covariant conservation of the matter perturbation stress-tensor alone unless the background is vacuum. In most of the literature on gravitational radiation reaction the background is assumed to be vacuum. This seems to be satisfactory for the Schwarzschild or Kerr metrics. But physically we have to deal not with an eternal black hole, but with the collapsing body which is not globally vacuous. Meanwhile the conservation laws which may hold in the asymptotically flat case has to be considered globally. It turns out that we must take into account contribution from the (perturbed) source of the background field as well. This is directly implied by the Bianchi identities.

5.1 Bianchi identity

Let the background be generated by the stress-tensor $\overset{B}{T}{}^{\mu\nu}$. We are interested by gravitational radiation emitted by a point particle of mass m moving along the geodesic of the background. Perturbations caused by the particle are assumed to be small so the particle stress-tensor

$$\overset{m}{T}{}^{\mu\nu} = m \int \dot{z}^\mu(\tau) \dot{z}^\nu(\tau) \frac{\delta(x - z(\tau))}{\sqrt{-g}} d\tau \quad (83)$$

is the first order quantity with respect to $T^{\mu\nu}$. By construction, the tensor (83) is divergence-free provided the particle follows the geodesic in the space-time. Since it is the first order quantity, it must be divergence-free with respect to the background covariant derivative up to the terms of the second order. Expanding the full metric $\hat{g}_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}$, where $\kappa^2 = 8\pi G$, and the Einstein tensor

$$\hat{G}^{\mu\nu} = G^{\mu\nu} + {}^1G^{\mu\nu}, \quad (84)$$

we could naively expect the full Einstein equations to be of the form

$$\hat{G}^{\mu\nu} = \kappa^2 \left(T^{\mu\nu} + {}^1T^{\mu\nu} \right). \quad (85)$$

Since $G^{\mu\nu} = \kappa^2 T^{\mu\nu}$, we then should have

$${}^1G^{\mu\nu} = \kappa^2 {}^mT^{\mu\nu}. \quad (86)$$

But the left hand side of this equation is not divergence-free with respect to the background covariant derivative. Expanding the full covariant derivative as

$$\hat{\nabla}_\mu = \nabla_\mu + \nabla_\mu \quad (87)$$

and taking into account the Bianchi identity for the background $\nabla_\mu G^{\mu\nu} = 0$ we obtain in the first order

$$\nabla_\mu {}^1G^{\mu\nu} = -\nabla_\mu G^{\mu\nu}. \quad (88)$$

Therefore

$$\nabla_\mu {}^mT^{\mu\nu} = -\nabla_\mu T^{\mu\nu}, \quad (89)$$

where the right hand side is the first order quantity. Thus Eq. (86) is contradictory. Physically the reason is that we have to take into account the perturbation of the background $\delta T^{\mu\nu}$ caused by the particle, so that the correct equation should be

$${}^1G^{\mu\nu} = \kappa^2 \left(T^{\mu\nu} + \delta T^{\mu\nu} \right). \quad (90)$$

But this is not an equation for $h_{\mu\nu}$, since in order to find $\delta T^{\mu\nu}$ one has to consider the matter field equations for the background metric. The problem thus becomes essentially non-local. This non-locality does not reduce to that of the tail term in the DeWitt-Brehme equation.

5.2 Vacuum background

If $T^{\mu\nu} = 0$ the above obstacle is removed and one can proceed further with the linearized equations for $h_{\mu\nu}$. The derivation of the reaction force initiated in [27]

and completed in [23] was based on the DeWitt-Brehme type calculation involving the integration of the field momentum over the small tube surrounding the particle world-line. As was noted later [28], this derivation had some drawbacks (for more recent discussion and further references see [29]). One problem consisted in computing the contributions of "caps" at the ends of the chosen tube segment which were not rigorously calculated. Another problem was the singular integral over the internal boundary of the tube which was simply discarded. In addition, the usual mass-renormalization is not directly applicable in the gravitational case: due to the equivalence principle the mass does not enter into the geodesic equations.

In [17] the local derivation of the gravitational reaction force was given which is free from the above problems and, in addition, is much simpler technically. It deal with the quantities defined only on the world line and does not involve the ambiguous volume integrals over the world-tube at all. The elimination of divergences amounts to the redefinition of the affine parameter on the world-line.

We start with reparametrization invariant form of the particle action introducing the einbein $e(\tau)$ on the world line acting as a Lagrange multiplier:

$$S[z^\mu, e] = -\frac{1}{2} \int \left[e(\tau) g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu + \frac{m^2}{e(\tau)} \right] d\tau. \quad (91)$$

Variation with respect to $z^\mu(\lambda)$ and $e(\tau)$ gives the equations

$$\frac{D}{d\tau}(e\dot{z}^\mu) = 0, \quad e = \frac{m}{\sqrt{-g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu}}, \quad (92)$$

and we obtain the geodesic equation in a manifestly reparametrization invariant form

$$\frac{D}{d\tau} \left(\frac{\dot{z}^\lambda}{\sqrt{-g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu}} \right) = 0. \quad (93)$$

Assuming now that the particle motion with no account for radiation reaction is geodesic on the background metric, the perturbed equation in the leading order in κ will read

$$\ddot{z}^\mu = \frac{\kappa}{2} \left(g^{\mu\nu} - \frac{\dot{z}^\mu \dot{z}^\nu}{\dot{z}^2} \right) (h_{\lambda\rho;\nu} - 2h_{\nu\lambda;p}) \dot{z}^\lambda \dot{z}^p, \quad (94)$$

where contractions are with the background metric.

The particle energy momentum tensor in our formulation will read

$$\overset{m}{T}^{\mu\nu} = \int e(\tau) \dot{z}^\mu(\tau) \dot{z}^\nu(\tau) \frac{\delta(x - z(\tau))}{\sqrt{-g}} d\tau, \quad (95)$$

and we choose the non-perturbed ein-bein $e_0 = \text{const}$ as a bare parameter. After the calculation similar to that of the preceding section we obtain

$$\ddot{z}^\mu = \kappa^2 e_0 \left\{ \frac{7}{2\varepsilon} \ddot{z}^\mu + \frac{1}{4} \Pi^{\mu\nu} \int_{-\infty}^{\tau} \left[4v_{\nu\lambda\alpha\beta;p} - 2(g_{\nu\lambda} v_{\sigma\alpha\beta;p}^\sigma + v_{\lambda\rho\alpha\beta;\nu}) - g_{\lambda\rho} v_{\sigma\alpha\beta;\nu}^\sigma \right] \dot{z}'^\alpha \dot{z}'^\beta \dot{z}'^\lambda \dot{z}'^\rho d\tau' \right\}. \quad (96)$$

Renormalization of the einbein is

$$\left(\frac{1}{e_0} - \frac{7\kappa^2}{2\varepsilon} \right) \ddot{z}^\mu = \frac{1}{e} \ddot{z}^\mu. \quad (97)$$

Finally we choose the renormalized affine parameter so that $\dot{z}^2 = -1$, which is equivalent to set $e = m$ and obtain the MiSaTaQuWa equation. As was shown in [17] this equation remains valid in a class on non-vacuum metrics, in particular, for Einstein spaces.

5.3 Gravitational radiation for non-geodesic motion

If the particle world-line is non-geodesic, the radiation reaction force contains a putative antidamping term which is a local part of the *rad* contribution to the self-force:

$$f_{\text{rad}}^\mu = -\frac{11\kappa^2}{3} (g^{\mu\nu} - \dot{z}^\mu \dot{z}^\nu) \ddot{z}_\nu + \text{tail}. \quad (98)$$

The reason is simply that the source of gravitational radiation is incomplete and the stress tensor is not divergence-free as required. Indeed, if the force is non-gravitational, one has to take into account the contribution of stresses of the field causing the body to accelerate. For instance, to describe gravitational radiation of an electron in the atom, one has to add the contribution from the Maxwell field stresses (spatial components, non-relativistic motion):

$$\square \psi^{ij} = -\kappa^2 G T^{ij}, \quad T^{ij} = T_{ij}^m + T_{ij}^s, \quad (99)$$

where

$$T_{ij}^m = \sum_{a=1,2} m_a \dot{z}_a^i \dot{z}_a^j \delta^3(X_a), \quad T_{ij}^s = -\frac{e_1 e_2}{4\pi} \frac{X_1^i X_2^j}{(X_1^2 X_2^2)^{3/2}} + (i \leftrightarrow j), \quad (100)$$

and $X_a^i = x^i - z_a^i(t)$. Using this source one can calculate the gravitational force and find the gravitational Schott term

$$f_{\text{Gshott}}^i = -\frac{G\mu}{15} \frac{d^5 D^{ij}}{dt^5} x_j, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (101)$$

where D^{ij} is the quadrupole moment.

Note that this derivation of the Schott term is not based on the local calculation, the two-body treatment was necessary. This features seems to be general: gravitational radiation reaction from non-geodesically moving particle can not be described by some DeWitt-Brehme-type equation.

6 Conclusions

The purpose of this lecture was to discuss some subtle points associated with interpretation of the radiation reaction force. We have shown that the Lorentz-Dirac equation in classical electrodynamics describes the balance of three and not just two momenta: the mechanical momentum of the particle, the momentum of emitted radiation, and the momentum carried by the electromagnetic field bound to the charge. The total momentum is conserved, but this does not imply an instantaneous balance of the emitted momentum and that of the particle. The bound field momentum described by the Schott terms destroys the local balance. The total balance, however, is restored if one consider the situation when the charge has zero acceleration at the initial and final moments, or for a periodic motion subject to averaging. These considerations are equally applicable to radiation reaction of a charge in curved space-time and for gravitational radiation reaction. This explains, in particular, the necessity of averaging in calculating the evolution of the Carter constant in the Kerr field [16].

A novel feature related to curved space is the existence of the finite WWSW force arising due to the tidal deformation of the bound electromagnetic field of the charge. This force is often interpreted as part of radiation reaction force, but one has clearly understand, however, that it has nothing to do either with radiation or with the Schott force. As we have shown, it is given by the T-even part of the retarded field, and thus present a finite remnant from the mass renormalization.

Derivation of the reaction force of non-gravitational nature acting on a charge moving (both geodesically and non-geodesically) in curved space-time can be computed directly substituting the retarded field into the equations of motion, as is the Minkowski space. The regularization is easily achieved by the point-splitting, and divergences are eliminated by renormalization of mass. In higher dimensions one needs counterterms depending on higher derivatives of the velocity. Divergencies may contain the Riemann tensor of the background.

Gravitational radiation reaction force can be obtained in a way similar to non-gravitational one only in vacuum space-time. In non-vacuum background the source of radiation apart from the local contribution from the particle must contain the contribution from the perturbed background. This can be seen from the analysis of the Bionchi identity. This second contribution is non-local, so the possibility to obtain the equation of the DeWitt-Brehme type seems implausible.

For a non-geodesically moving mass the formal derivation of the reaction force leads to putative antidamping effect. To cure this problem one has to take into account the contribution of stresses forcing the mass to accelerate. Then in the non-

relativistic case one derives the gravitational quadrupole Schott term, but the derivation is non-local. This is another example when the (quasi)local equation of motion with the reaction force does not exist. Here by quasilocality we mean the possibility of the tail term.

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